Noncommutative geometry

an introduction

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C^*-algebras

- A C^*-algebra A is a Banach algebra over $\mathbb{C}$, together with an involution $* : A \to A$ satisfying:

$$(x + y)^* = x^* + y^*$$
$$(xy)^* = y^* x^*$$
$$(\lambda x)^* = \bar{\lambda} x^*$$
$$||x^* x|| = ||x^*||\,||x||$$

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$$

Example

$M_n(\mathbb{C})$ with $*$=conjugate transpose and with operator norm.

Motivation

Theorem

a) For any commutative C^*-algebra A with spectrum $\hat{A}$ the Gelfand transform $A \to C_0(\hat{A})$ is a $*$-isomorphism
b) Any C^*-algebra is ($*$-isomorphic to) a C^*-subalgebra of the algebra $\mathbb{L}(H)$ of bounded operators on some Hilbert space H [Gelfand-Naimark].

Noncommutative tori

- Consider a foliation F of torus V given in local coordinates by:
$$dy = \theta dx$$
with $\theta \in \mathbb{R}/\mathbb{Q}$. All its leaves are homeomorphic to $\mathbb{R}$ and quotient topology of the leaf space X is coarse; thus $L^p(X, \mathbb{C}) = \mathbb{C}$.

Random operators

- Let $L^2(l)_x, x \in V$ be the bundle of half-densities on leaves. Then we call q_l a random operator if for all measurable sections η_x, ζ_x of this bundle mapping $x \to \langle q_l\, \eta_x, \zeta_x \rangle \in \mathbb{C}$ is measurable.

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Bounded Borel function acting by multiplication.

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Let X be a real vector field tangent to the leaves of foliation. Then $\psi_t = exp(tX)$ defines a family of diffeomorphisms of V and corresponding family of unitaries U_l acts as: $U_l\, \eta_x = \eta_{\psi_t(x)}$

von Neumann algebra of Kronecker foliaion

- von Neumann algebra of Kronecker foliation is $\{W_\theta(m,n); m,n \in \mathbb{Z}\}''$ where $W_\theta(m,n)\psi(t) = e^{-\pi i\theta mn}e^{2\pi i\theta nt}\psi(t-m)$. Double prime $''$ means closure in weak operator topology. There is a trace τ on $W(V,F)$ which is:

$$\tau(W_\theta(m,n)) := \begin{cases} 1, \text{ m=n=0} \\ 0 \ \textit{otherwise} \end{cases}$$

Classification of von Neumann algebras

Theorem

All factors (i.e. VN algebras with $Z(W) = \mathbb{C}$) can be divided into 3 types.
I) $M_n(\mathbb{C})$ and $\mathbb{L}(H)$ with H-infinite dimensional and separable. Achieved when $\lambda : V \to X$ admits a measurable section.
II) Algebras not of type I but having a positive semi-finite faithful normal trace.
III) Algebras with non-trivial time evolution $\sigma^{-it} M \sigma^{it} = M$ (the Anosov foliation on the Riemann surface of $g > 1$) [Murray-von Neumann, Tomita-Takesaki]

Swan's theorem

The functor

$$\{\text{Vector bundles on } X\} \to \{\text{Fin. gen. projective } C^\infty(X)\text{-modules}\}$$

given by

$$E \to \Gamma(E)$$

is an equivalence of categories. Say we have a finite projective module P over A. Then for some A-module we have $P \oplus Q \simeq A^n$ so that there is a projection $p \in M(n, A)$. In fact, finite projective modules are classified up to isomorphism by classes of matrix idempotents up to unitary equivalence ($e \sim f \Leftrightarrow e = ufu^{-1}$ for u unitary).

K-theory

- $K_0(R)$ where R is a unital ring is the group of differences $[E] - [F]$ where $[E], [F]$ are isomorphism classes of finite projective modules over R under operation of direct sum:$([E] - [F]) + ([A] - [B]) = ([E \oplus A] - [F \oplus B])$.

"Continuous" noncommutative tori

It's possible to show that at continuous (rather than measure-theoretic level) the algebra of functions on noncommutative torus is a universal C^*-algebra on two generators U, V which satisfy $UV = \lambda VU$, $\lambda \in \mathbb{C}$ (you can think of U and V as acting on $L^2(S^1)$ by $(Uf)(z) = zf(z)$, $(Vf) = f(\lambda z) = f(e^{2\pi i\theta}z)$). Its generic element is $\sum_{m,n\in\mathbb{Z}} a_{mn}U^mV^n$.

K-theory of noncommutative torus

A non-trivial idempotent in A_θ can be obtained by making an ansatz: $p = f \bullet V^{-1} + g + h \bullet V$ where $f, g, h \in C(S^1)$ (recall that $(Vq)(t) = q(\lambda t)$). By definition of projection we have $p = p^* = p^2$ giving us equations on functional parameters. These equations give rise to several equivalent projections. By results of Pimsner and Voiculescu generators of $K_0(A_\theta)$ are $[1] \in M_1(A_\theta) = A_\theta$ and (the equivalence class of) above projection. In particular, $K_0(A_\theta) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Cyclic cohomology

- Define $C^n(A) = Hom(A^{\otimes n+1}, \mathbb{C})$ and
  $(b\phi)(a_0, ..., a_{n+1}) =$
  $\sum_{i=0}^{n}(-1)^i f(a_0, ..., a_i a_{i+1}, .., a_n) + (-1)^{n+1}\phi(a_{n+1}a_0, ...a_n)$
  An n-cochain is called cyclic if $\phi(a_n, a_0, ..., a_{n-1}) = (-1)^n\phi(a_0, ..., a_n)$
  Denote the space of cyclic cochains as $C^n_\lambda(A)$. It can be shown that cyclicity is invariant under b so that we have a well-defined complex $(C^n_\lambda(A), b)$.

Example

Example

Take $A = \mathbb{C}$. Then $C_\lambda^{2n}(A) = \mathbb{C}$, $C_\lambda^{2n+1}(A) = 0$ hence cyclic complex looks like: $\mathbb{C} \to 0 \to \mathbb{C} \ldots$. So:

$$\mathrm{HC}^n(A) = \begin{cases} \mathbb{C}, n = 2k \\ 0, \text{ n=2k+1} \end{cases}$$

Hochschild cohomology of complex numbers vanishes in higher degrees hence mapping $I : HC^n(A) \to H^n(A, A^*)$ induced by inclusion of complexes $C_\lambda^n(A) \subset C^n(A)$ is not always injective.

Let $A = C^\infty(M)$ be the algebra of smooth complex-valued functions on smooth closed oriented manifold M. Let

$$\phi(f_0, ..., f_n) := \int_M f_0 df_1 \wedge ... \wedge df_n$$

be an n-cocycle. By straightforward computation we have $b\phi = 0$ and as $\int_M (f_n df_0 \wedge ... \wedge df_{n-1} - (-1)^n f_0 df_1 \wedge ... \wedge df_n) = \int_M d(f_n f_0 df_1 \wedge ... \wedge df_{n-1}) = 0$ ϕ is a cyclic n-cocycle. More generally, denote the space of de Rham p-currents by $\Omega_p M$. Then for any p-current C:

$$\phi_C(f_0, ..., f_p) := \langle C, f_0 df_1 \wedge ... \wedge df_p \rangle$$

is a Hochschild cocycle (that is $b\phi_C = 0$). If C is closed (that is for any $(p-1)$-form ω $\langle C, d\omega \rangle = 0$) then it's cyclic cocycle. We thus get mappings:

$$\Omega_m M \to H^m(C^\infty(M), C^\infty(M)^*),\ Z_m M \to HC^m(C^\infty(M))$$

where $Z_m M \subset \Omega_m M$ is the space of closed m-currents.

Connes-Chern pairing

Let ϕ be a cyclic $2n$-cocycle on algebra A. Then

$$\tilde{\phi}(m_0 \otimes a_0, ..., m_{2n} \otimes a_{2n}) = tr(m_0...m_{2n})\phi(a_0, ..., a_{2n})$$

defines cyclic cocycle on $M_k(\mathbb{C}) \otimes A = M_k(A)$. We now define pairing $HC^{2n}(A) \otimes K_0(A) \to \mathbb{C}$:

$$\langle [\phi], [e] \rangle = \frac{1}{n!}\tilde{\phi}(e, ..., e)$$

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Example

$HC^0(A)$ is the space of traces on A. So:

$$HC^0(A) \times K_0(A) \to \mathbb{C}$$

$$(\tau, e) = \sum \tau(e_{ii})$$

Idempotent conjecture

There is a conjecture due to Kadison and Kaplansky which states that for a discrete, countable, torsion-free group G $C_r^*(G)$ has no nontrivial idempotents. We have canonical trace $C_r^*(G) \to \mathbb{C}$ given by $\tau(\sum a_g g) = a_1$ and extended by continuity. It can be extended further to $\tau_* : K_0(C_r^*(G)) \to \mathbb{C}$. Up to conjugation we have $e = e^* = e^2$ so $\tau_*(e) = \tau_*(ee^*) \geq 0$ and analogously $\tau_*(1-e) = \tau_*((1-e)(1-e^*)) \geq 0$. As $\tau_*(e) + \tau_*(1-e) = 1$, if $Im\, \tau_* \subseteq \mathbb{Z}$ then $\tau_*(e) = 0$ (Baum-Connes conjecture known for discrete subgroups of $SO(n,1)$ and $SU(n,1)$, Gromov hyperbolic groups, etc. implies it).