Indices of commuting differential operators with meromorphic coefficients

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Motivation

- KP equation describing shallow water waves admits Lax presentation

$$\frac{\partial L}{\partial t} = [L,\ A], \qquad L = \frac{\partial}{\partial y} - \frac{\partial^2}{\partial x^2} + U(x,\ y, t),$$

$$A = \frac{\partial}{\partial t} - \frac{\partial^3}{\partial x^3} + \frac{3}{2} U \frac{\partial}{\partial x} - W(x, y, t).$$

The class of finite-gap solutions is singled out by requiring existence of ODO B

$$B = \sum_{i=0}^{k} w_i(x) \partial_x^i, \qquad [L_{t=0}, B] = 0.$$

Finite-gap solutions are generically quasi-periodic in x and t.

- Thus solition theory is naturally connected to study of commutative subalgebras of algebra of ordinary differential operators.

Algebraic geometry

- Schur [1] proved that if an operator L_n of order $n > 1$ commutes with operators L_m, L_k ($L_nL_m = L_mL_n$, $L_nL_k = L_kL_n$) then $L_kL_m = L_mL_k$. Namely, operators commuting with a given operator form a commutative ring (this is non-trivial as it's false over general graded rings).
- Burchnall and Chaundy [2] proved that $L_nL_m = L_mL_n$ implies that there's a polynomial $R \in \mathbb{C}[x, y]$ such that $R(L_n, L_m) = 0$. The completion of affine curve $R(x, y) = 0$ gives us a Riemann surface S called spectral curve.

Algebraic geometry

- Consider the space of common eigenfunctions $V_{v,w} = \{L_n\psi = z\psi, L_m\psi = w\psi, (z, w) \in S\}$. Its dimension at general point is called the rank r of the pair L_n, L_m. Typically we have $r = gcd(n, m)$.
- In the case $r = 1$ the problem the coefficients of L_n, L_m were expressed in terms of S theta-function by Krichever [3].
- In the case $r > 1$ the problem is equivalent to matrix Riemann-Hilbert factorization and cannot be solved with current methods. Partial results were obtained in [4], [5] (for S an elliptic curve).

Operators with meromorphic coefficients, I

- Suppose we have $L = \Sigma_{i=1}^{n} a_i(z)\partial^{n-i}$ with $a_i(z) = b_i z^{-i}(1 + O(z))$. Let $L' = \Sigma_{i=1}^{n} b_i z^{-i}\partial^{n-i}$. Then $L'(z^m) = z^m * P(m)$ where P is a polynomial in m. Its n roots m_j are called *indices* of L.
- For $r = 1$ the indices are integers distinct mod n [6].
- A natural question to ask: what are indices when $r > 1$?

Operators with meromorphic coefficients, II

Theorem

Indices of Mironov's operators [7] lie in $\mathbb{Q}(\sqrt{4k+\sqrt{4m+1}})$ *for some integers* k, m.

Theorem

Indices of Zuo's operator [8] are integral.

Operators with meromorphic coefficients, III

Conjecture

Pair of commuting operators L, A has integral indices if and only if it can be smoothly deformed into degenerate pair, i.e. $(M_1^k, M_2^k$ where M_1, M_2 commute with rank 1.

- Can one construct *effective* classification scheme for deformation classes of commuting operators with meromorphic coefficients in terms of indices?

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