Lower estimates for the energy functional on a family of Hamiltonian minimal Lagrangian tori in $\mathbb{C}P^2$

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Lagrangian submanifolds

- Let (X, ω) be a symplectic manifold. An immersion $f : \Sigma \to X$ is called Lagrangian iff $f^*\omega = 0$ and $\dim \Sigma = \frac{1}{2} \dim X$.

Lagrangians in $\mathbb{C}P^2$

- Let $\mathcal{H} : S^5 \subset \mathbb{C}^3 \to \mathbb{C}P^2$ be the Hopf projection. Examples of Lagrangian submanifolds in $\mathbb{C}P^2$:
  1. Homogeneous tori:
     $\Sigma_{r_1,r_2,r_3} = \left\{ \mathcal{H}(z_1, z_2, z_3) \middle| |z_1|^2 = r_1^2, |z_2|^2 = r_2^2, |z_3|^2 = r_3^2 \right\}$ with $r_1^2 + r_2^2 + r_3^2 = 1$.
  2. Totally geodesic $\mathbb{R}P^2 = \left\{ \mathcal{H}(z_1, z_2, z_3) \middle| z_i = \bar{z}_i, 1 \leq i \leq 3 \right\}$.
- Non-examples:
  1. There are no embedded closed orientable Lagrangian surfaces of non-zero Euler characteristic in $\mathbb{C}P^2$ (Seidel, Mohnke). For explicit examples of immersions see Castro-Urbano, Audin.
  2. There are no embedded Lagrangian Klein bottles in $\mathbb{C}P^2$ (Nemirovski, Shevchishin). For explicit examples of immersions see Mironov.

Specialty conditions for Lagrangian tori

- Let $\Sigma \subset \mathbb{C}P^2$ be a Lagrangian torus. The Maslov 1-form $\alpha_H = \omega(H, \cdot)$ is closed (Dazord).
- $\alpha_H = 0$ for minimal Σ. Lawson-Simons proved that any minimal stable submanifold of $\mathbb{C}P^n$ is complex thus minimal Lagrangians are never stable.
  Example: the Clifford torus $\Sigma_{Cl} = \Sigma_{\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}}$.
- $\delta\alpha_H = 0$ for Hamiltonian minimal Σ, i.e. Σ is the critical point of the volume functional under Hamiltonian deformations ($\iota_v\omega$ is exact).
  Example: any homogeneous torus Σ_{r_1,r_2,r_3}.

Lagrangian tori and 2D Schrödinger equation

- Let $\beta : U \to \mathbb{R}$ be a local function satisfying $d\beta = \alpha_H$. Choose conformal coordinates and pass to the universal cover of Lagrangian torus $r : \mathbb{R}^2 \to S^5 \subset \mathbb{C}^3$. Mironov has noticed that r satisfies 2D Schrödinger equation

$$Lr = 0, \qquad L = (\partial_x - \frac{i\beta_x}{2})^2 + (\partial_y - \frac{i\beta_y}{2})^2 + V(x,y),$$

$$V = 2g + \frac{1}{4}(\beta_x^2 + \beta_y^2) + \frac{i}{2}\Delta\beta$$

where g is the conformal factor of the induced metric.

Energy functional for Lagrangian tori

- The integral of the potential over the fundamental domain Λ is called the energy functional

$$E(\Sigma) = \frac{1}{2} \int_{\Lambda} V \, dx \wedge dy.$$

- $$E(\Sigma) = A(\Sigma) + \frac{1}{8} W(\Sigma),$$

$$A(\Sigma) = \int_{\Sigma} d\sigma, \qquad W(\Sigma) = \int_{\Sigma} |H|^2 d\sigma,$$

where $d\sigma$ is the induced area element (Ma-Mironov-Zuo).

The energy conjecture

- Ma-Mironov-Zuo have conjectured that

$$E(\Sigma) \geq E(\Sigma_{Cl}) = \frac{4\pi^2}{3\sqrt{3}}$$

for any Lagrangian torus $\Sigma \subset \mathbb{C}P^2$.

Partial results

- Ma-Mironov-Zuo have proved the conjecture for homogeneous tori Σ_{r_1,r_2,r_3}.
- Haskins' results imply the conjecture for minimal Lagrangian tori of sufficiently large spectral genus.
- Goldstein has established the inequality

$$E(\Sigma) \geq \frac{3}{\pi} E(\Sigma_{Cl})$$

for any Lagrangian torus Hamiltonian isotopic to the Clifford torus.

Kinematic formula & Floer cohomology

- For compact Lagrangian submanifolds $\Sigma_1, \Sigma_2 \subset \mathbb{C}P^n$ we have

$$\mathrm{vol}(\Sigma_1)\mathrm{vol}(\Sigma_2) = \frac{1}{c_n} \int_{SU(n+1)} \#(g\Sigma_1 \cap \Sigma_2) dg$$

  with c_n depending only on n (Howard).
- For Σ Hamiltonian isotopic to Σ_{Cl} we have $\#(g\Sigma \cap \Sigma) \geqslant 4$ (Cho). Therefore

$$\mathrm{vol}(\Sigma)^2 \geqslant 4\frac{\mathrm{vol}(SU(3))}{c_2} = \frac{16\pi^2}{3}.$$

- The mapping $\psi : \mathbb{R}^2 \to \mathbb{C}P^2$

$$\psi(x,y) = (F_1(x)e^{i(G_1(x)+\alpha_1 y)} : F_2(x)e^{i(G_2(x)+\alpha_2 y)} : F_3(x)e^{i(G_3(x)+\alpha_3 y)}),$$

$$F_i = \sqrt{\frac{g(x)+\alpha_{i+1}\alpha_{i+2}}{(\alpha_i-\alpha_{i+1})(\alpha_i-\alpha_{i+2})}}, \qquad G_i = \frac{\alpha_i}{2}\int_0^x \frac{2c_2 - ag(z)}{\alpha_i g(z) - c_1}dz,$$

$$g(x) = a_1\left(1 - \frac{a_1-a_2}{a_1}\mathrm{sn}^2\left(x\sqrt{a_1+a_3}, \frac{a_1-a_2}{a_1+a_3}\right)\right)$$

is a conformal Hamiltonian minimal Lagrangian immersion, where $a_1 > a_2 > 0, \alpha_i \in \mathbb{R}$ and the rest of the constants can be expressed in terms of a_i, α_i. If rationality conditions are met, its image is a torus Σ_M invariant under S^1-group of ambient isometries (Mironov, Ma).

Results

- Following inequality holds (K.)

$$E(\Sigma_M) > E(\Sigma_{Cl}).$$