ÉTALE COHOMOLOGY

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Abstract. In these notes we explain the construction and some of the properties of étale cohomology and etale homotopy type. We do not describe étale cohomology with compact supports, Poincaré duality, the Gysin sequence or l-adic cohomology.

1. Introduction

Let $X \to \operatorname{Spec} F_q$ be a smooth n-dimensional projective variety over the field with q elements. Define its zeta function ζ_X as follows

$$\zeta_X(s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m} q^{-ms}\right)$$

where N_m is the number of points of $X \times_{\operatorname{Spec} F_q} \operatorname{Spec} F_{q^m}$.

André Weil has proposed the following conjectures:

(1) if we set $T = q^{-s}$, then ζ_X has the following form

$$\zeta_X(s) = \frac{P_1(T) \dots P_{2n-1}(T)}{P_0(T) \dots P_{2n}(T)},$$

where $P_0(T) = 1 - T$, $P_{2n}(T) = 1 - q^n T$, and for $1 \leq i \leq 2n-1$ $P_i(T) = \prod_j (1 - \alpha_{ij} T)$ for some α_{ij}.

(2) Zeta function satisfies the functional equation

$$\zeta_X(n-s) = \pm q^{\frac{nE}{2} - Es} \zeta_X(s).$$

(3) $|\alpha_{ij}| = q^{\frac{i}{2}}$ for all j and $1 \leq i \leq 2n-1$. This particular conjecture is sometimes referred to as Riemann's hypothesis.

(4) If X is a mod p reduction of a smooth projective variety defined over a number field embedded in $\mathbb{C}$, then the degree of $P_i(T)$ is equal to the i-th Betti number of $X(\mathbb{C})$, the set of complex points of $\mathbb{C}$ endowed with complex-analytic topology.

Since the points of $X \times_{\operatorname{Spec} F_q} \operatorname{Spec} F_{q^m}$ can be interpreted as fixed points of the m-th power of Frobenius endomorphism, the conjectures would follow from existence of a "reasonable" cohomology theory for smooth projective varieties over F_q by application of a Lefschetz-type trace formula.

However, it is not entirely obvious how to construct the necessary cohomology theory. For example, in characteristic p there exist smooth elliptic curves whose endomorphism ring is an order in a quaternion algebra over $\mathbb{Q}$. This should act on the first cohomology group of the elliptic curve, which we would expect to be a vector field of dimension 2. This shows that the coefficient field for Weil cohomology

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theory can not possibly be $\mathbb{Q}$. One can similarly see that p-adic numbers can not be the coefficient field. It turns out, however, that the necessary theory exists over l-adic numbers for $l \neq p$. We will not explicitly describe this cohomology theory here but rather we will construct its essential building block: étale cohomology of torsion sheaves.

Remark 1.1. Strictly speaking, the above argument only applies in the case of ground field containing F_{p^2} for some prime p. The reason is that the endomorphism algebra of a supersingular elliptic curve defined over the prime field F_p never has rank 4—it is necessary to change the base field (and in fact, it is enough to extend only to F_{p^2}, see e.g. [9]).

Therefore, this still leaves open the possibility of $\mathbb{Q}_p$-vector space valued Weil cohomology theory for smooth projective varieties defined over the prime field F_p. The so-called crystalline cohomology, defined over the ring of Witt vectors $W(F_p)$, provides an example of such a cohomology theory (after taking tensor product with $Frac(W(F_p)) \approx \mathbb{Q}_p$).

2. Construction of étale cohomology

One might think that to define the necessary cohomology theory one might simply take the derived functor cohomology of the constant sheaf associated to $\mathbb{Z}/l$ (for l prime distinct from the characteristic of the field of definition of X). However, this does not work—for an irreducible scheme X, the constant sheaf is flasque so its higher cohomology vanishes (so in particular, the property (2) from the introduction can not work).

This problem can be solved if we take sheaf cohomology with respect to a different topology on our scheme X. This topology is called étale topology and it is intended to imitate the analytic topology usually considered in the theory of complex-analytic spaces. It should be noted that étale topology is not really a topology in the point-set sense; rather, it is what is called a Grothendieck topology.

2.1. Sites in general.

Definition 2.2. Let $\mathcal{C}$ be a category. A *family of morphisms with fixed target* $\mathcal{U} = \{\varphi_i : U_i \to U\}_{i \in I}$ is the data of

(1) an object $U \in \mathcal{C}$,
(2) a set I (possibly empty), and
(3) for all $i \in I$, a morphism $\varphi_i : U_i \to U$ of $\mathcal{C}$ with target U.

Definition 2.3. A *site* consists of a category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ consisting of families of morphisms with fixed target called *coverings*, such that

(1) (isomorphism) if $\varphi : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{\varphi : V \to U\}$ is a covering,
(2) (locality) if $\{\varphi_i : U_i \to U\}_{i \in I}$ is a covering and for all $i \in I$ we are given a covering $\{\psi_{ij} : U_{ij} \to U_i\}_{j \in I_i}$, then
$$\{\varphi_i \circ \psi_{ij} : U_{ij} \to U\}_{(i,j) \in \prod_{i \in I} \{i\} \times I_i}$$
is also a covering, and
(3) (base change) if $\{U_i \to U\}_{i \in I}$ is a covering and $V \to U$ is a morphism in $\mathcal{C}$, then
  (a) for all $i \in I$ the fibre product $U_i \times_U V$ exists in $\mathcal{C}$, and

(b) $\{U_i \times_U V \to V\}_{i \in I}$ is a covering.

It is easy to see that any topological space gives rise to a site: open subsets are the objects, morphisms are inclusions and coverings are surjective families of morphisms.

Definition 2.4. A presheaf $\mathcal{F}$ of sets (resp. abelian presheaf) on a site $\mathcal{C}$ is a *sheaf* if for all coverings $\{\varphi_i : U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$, the diagram

$$\mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j), \tag{2.4.1}$$

where the first map is $s \mapsto (s|_{U_i})_{i \in I}$ and the two maps on the right are $(s_i)_{i \in I} \mapsto (s_i|_{U_i \times_U U_j})$ and $(s_i)_{i \in I} \mapsto (s_j|_{U_i \times_U U_j})$, is an equalizer diagram in the category of sets (resp. abelian groups).

Obviously, this definition reproduces the classifical definition of sheaf for site we associated above to topological spaces.

Definition 2.5. We denote $\mathit{Sh}(\mathcal{C})$ (resp. $\mathit{Ab}(\mathcal{C})$) the full subcategory of $\mathit{PSh}(\mathcal{C})$ (resp. $\mathit{PAb}(\mathcal{C})$) whose objects are sheaves. This is the *category of sheaves of sets* (resp. *abelian sheaves*) on $\mathcal{C}$.

Remark 2.6. The category of abelian sheaves is not an abelian subcategory of the the category of abelian presheaves (if we endow the two with the standard abelian structures).

It is well-known that the category of abelian sheaves on a site is an abelian category with enough injectives. Therefore we can define cohomology as the right-derived functors of the sections functor (which is manifestly left exact). If $U \in \mathrm{Ob}(\mathcal{C})$ and $\mathcal{F} \in \mathit{Ab}(\mathcal{C})$,

$$H^p(U, \mathcal{F}) := R^p\Gamma(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$$

where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution.

2.7. **Étale site.**

Definition 2.8. Let $R \to S$ be a ring map. We say S is *formally smooth over* R if for every commutative solid diagram

$$\begin{array}{ccc} S & \longrightarrow & A/I \\ \uparrow & \searrow & \uparrow \\ R & \longrightarrow & A \end{array}$$

where $I \subset A$ is an ideal of square zero, a dotted arrow exists which makes the diagram commute.

A finitely presented map of rings that is formally smooth is called a smooth map. A smooth map $\phi : S \to R$ is called étale if the map of free S-modules

$$\bigoplus_{(a,b) \in S^2} S[(a,b)] \oplus \bigoplus_{(f,g) \in S^2} S[(f,g)] \oplus \bigoplus_{r \in R} S[r] \longrightarrow \bigoplus_{a \in S} S[a]$$

defined as follows

$$[(a,b)] \longmapsto [a+b] - [a] - [b], \quad [(f,g)] \longmapsto [fg] - f[g] - g[f], \quad [r] \longmapsto [\varphi(r)]$$

is surjective.

The definition of étale map of rings can be globalized fairly easily.

Definition 2.9. A map of schemes $f : X \to S$ is called étale if for every point $x \in X$ there an affine open neighbourhood U and an affine open $V \subset S$ with $f(U) \subset V$ such that the induced map between the rings of global functions on V and U is étale.

Lemma 2.10. *Let T be a scheme.*

(1) *If $T' \to T$ is an isomorphism then $\{T' \to T\}$ is an étale covering of T.*
(2) *If $\{T_i \to T\}_{i \in I}$ is an étale covering and for each i we have an étale covering $\{T_{ij} \to T_i\}_{j \in J_i}$, then $\{T_{ij} \to T\}_{i \in I, j \in J_i}$ is an étale covering.*
(3) *If $\{T_i \to T\}_{i \in I}$ is an étale covering and $T' \to T$ is a morphism of schemes then $\{T' \times_T T_i \to T'\}_{i \in I}$ is an étale covering.*

This lemma implies that the set of étale coverings of a given scheme forms a site, the so-called big étale site. Étale cohomology groups of a sheaf on big étale site can be defined as right-derived functors of the global sections.

To better understand this definition, we might ask what are the "points" of étale site.

Definition 2.11. Let $\mathcal{C}$ be a site. A *point p of the site $\mathcal{C}$* is given by a functor $u : \mathcal{C} \to Sets$ such that

(1) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ the map $\coprod u(U_i) \to u(U)$ is surjective.
(2) For every covering $\{U_i \to U\}$ of $\mathcal{C}$ and every morphism $V \to U$ the maps $u(U_i \times_U V) \to u(U_i) \times_{u(U)} u(V)$ are bijective.
(3) The stalk functor $Sh(\mathcal{C}) \to Sets$, $\mathcal{F} \mapsto \mathcal{F}_p$ is left exact.

If the reader has not seen this definition before, he is invited to check that for Zariski site it coincides with the usual definition of points of topological space.

Remark 2.12. More generally, it can be shown that in a topological space where every irreducible subset has a unique generic point (also known as a sober space), such as a normal Hausdorff space or the underlying space of a scheme, every abstract point of the category of sheaves on the topological space comes from a unique classical point.

Grothendieck has proved that every point of étale site of a scheme X comes from a morphism $\mathrm{Spec}(k) \to X$ where k is an algebraically closed field. Points of étale site are frequently referred to as *geometric points*.

We define the *stalk* of a presheaf $\mathcal{F}$ at p as

$$\mathcal{F}_p = \mathrm{colim}_{\{(U,x)\}^{opp}} \mathcal{F}(U). \tag{2.12.1}$$

The colimit is over the opposite of the category of neighbourhoods of p.

It can be shown that the stalk of the structure sheaf at a geometric point is a strictly henselian local ring (i.e. a local ring with a separably algebraically closed residue field such that Hensel's lemma holds).

3. Properties of étale cohomology

Assume we have a commutative diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f} \\ S' & \xrightarrow{g} & S \end{array}$$

Then for any étale sheaf on X and any $q \geq 0$ we have a natural morphism

$$g^* R^i f_* F \to R^i f'_* g'^* F.$$

Theorem 3.1. *If F is a torsion sheaf and f is a proper morphism, the natural morphism above is an isomorphism.*

Theorem 3.2. *If f is quasi-compact and quasi-separated, $S' = \varprojlim S_\lambda$ with $g_\lambda : S_\lambda \to S$ smooth and $S_{\lambda'} \to S_\lambda$ affine, and F is a torsion sheaf with torsion orders relatively prime to p, the natural morphism above is an isomorphism.*

Remark 3.3. Suppose K/k is an extension of separably closed fields, X is a quasi-compact quasi-separated k-scheme, and F is a torsion sheaf with torsion orders not divisible by the characteristic to k. Then, the base change map $H^i(X, F) \to H^i(X_K, F_K)$ is an isomorphism for $i \geq 0$.

Proof. From functoriality of base change we get a commutative diagram

$$\begin{array}{ccc} H^i(X,F) & \longrightarrow & H^i(X_K, F_K) \\ \downarrow & & \downarrow \\ H^i(X_{\bar{k}}, F_{\bar{k}}) & \longrightarrow & H^i(X_{\bar{K}}, F_{\bar{K}}) \end{array}$$

The vertical maps are isomorphisms since the morphisms $X_{\bar{k}} \to X$, $X_{\bar{K}} \to X_K$ are integral radicial surjections (and it can be shown that such morphisms induce equivalences on étale sites). The lower horizontal map is an isomorphism by the theorem above (the extension $\bar{K}/\bar{k}$ is a limit of smooth $\bar{k}$-algebras).

Remark 3.4. The divisibility condition on the torsion order in the above theorem is essential (as can be seen by considering the cohomology of a constant sheaf on the affine line over an algebraically closed field and over its non-trivial extension).

Let X be a scheme of finite type over a separably closed field k. Recall that a sheaf of abelian groups on X is called constructible if it is a Noetherian object of the category of sheaves of abelian groups. It turns out that for constructible sheaves, étale cohomology groups are especially well-behaved.

Theorem 3.5. *Assume that X is separated. Let F be a constructible torsion sheaf whose torsion orders are relatively prime to $char(k)$. Then $H^i_{et}(X, F)$ are finite for any $i \geq 0$.*

Theorem 3.6. *Assume that X is proper. Then for any constructible torsion sheaf F the groups $H^i_{et}(X, F)$ are finite for any $i \geq 0$.*

Étale cohomological dimension of schemes (resp. affine schemes) behaves similarly to the singular cohomological dimension of complex-analytic spaces (resp. Stein manifolds).

Theorem 3.7. *For any torsion sheaf F, we have $H^i_{et}(X, F) \approx 0$ for $i > 2\dim X$. If X is affine, the same holds for $i > \dim X$.*

4. Čech cohomology

One of the ways of computing étale cohomology is picking an étale cover and considering Čech complex.

Definition 4.1. Let $\mathcal{C}$ be a category, $\mathcal{U} = \{U_i \to U\}_{i\in I}$ a family of morphisms of $\mathcal{C}$ with fixed target, and $\mathcal{F} \in PAb(\mathcal{C})$ an abelian presheaf. We define the *Čech complex* $\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})$ by

$$\prod_{i_0\in I} \mathcal{F}(U_{i_0}) \to \prod_{i_0,i_1\in I} \mathcal{F}(U_{i_0} \times_U U_{i_1}) \to \prod_{i_0,i_1,i_2\in I} \mathcal{F}(U_{i_0} \times_U U_{i_1} \times_U U_{i_2}) \to \dots$$

where the first term is in degree 0, and the maps are the usual ones. Note that we allow indices to coincide. The *Čech cohomology groups* are defined by

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(\check{\mathcal{C}}^\bullet(\mathcal{U}, \mathcal{F})).$$

Artin has shown that étale cohomology and étale Čech cohomology agree for any torsion sheaf for a quasi-compact scheme such that any finite subset is contained in an affine open (note that for normal integral separated schemes of finite type over an algebraically closed field the latter condition equivalent to quasi-projectivity [1]). It has been conjectured [2] that étale cohomology and étale Čech cohomology agree for quasi-compact separated schemes.

5. Étale fundamental group

Having constructed étale cohomology groups, we might wonder whether there is a functorial construction of a homotopy type underlining them. In this section describe the construction of étale fundamental group; the construction of higher homotopy groups is somewhat more involved.

For a scheme X, denote the category of finite étale covers by FEt_X. For any geometric point $\bar{x} \in X$, we have a functor

$$F_{\bar{x}} : FEt_X \to \text{FinSet}, \qquad F_{\bar{x}}(Y) = Y_{\bar{x}}.$$

The étale fundamental group $\pi_1(X, \bar{x})$ of X at base point $\bar{x}$ is defined as the group of natural automorphisms of $F_{\bar{x}}$. There is actually a natural topology on this group; namely, we have a canonical injection $Aut(F_{\bar{x}}) \to \prod_{Y\in Obj(FEt_X)} Aut(F_{\bar{x}}(Y))$. It can be shown that if we endow each group in the product with discrete topology, $Aut(F_{\bar{x}})$ becomes a closed subgroup of the product. So $\pi_1(X, \bar{x})$ is naturally a profinite group.

It is fairly easy to see that for any two geometric points $\bar{x}$, $\bar{y}$ the functors $F_{\bar{x}}$ and $F_{\bar{y}}$ are (non-canonically) isomorphic. The induced isomorphism between π_1's is canonical up to inner automorphisms.

Let X be a $\mathbb{C}$-scheme locally of finite type. A version of Riemann's existence theorem provides an equivalence between the category of finite étale covers of X and the category of finite-degree topological coverings of $X(\mathbb{C})$ (endowed with complex-analytic topology). We can see therefore that our definition of étale fundamental group above is not totally unreasonable.

Example 5.1. Let K be a field. Fix an embedding $K \to \bar{K}$ and denote x : $\text{Spec}\,\bar{K} \to \text{Spec}\,K$ the corresponding geometric point. Let K^{sep} be the separable closure of K. Then there is an isomorphism $Gal(K^{sep}/K) \to \pi_1(\text{Spec}\,K, x)$ of profinite topological groups.

Example 5.2. Let X be a connected smooth projective curve of genus g over an algebraically closed field of characteristic $p > 0$. Then there is a surjection

$\phi : \widehat{F_{2g}} \to \pi_1(C)$, where $\widehat{F_{2g}}$ is the profinite completion of the free group on $2g$ elements. Moreover, if we let $G^{(p)}$ be the prime-to-p quotient of G, then

$$\pi_1(C)^{(p)} \cong \widehat{F_{2g}/w}^{(p)},$$

where $w = \prod_{i=1}^{g} [x_i, y_i]$.

The reader could wonder why do we include only finite étale covers in the definition of étale fundamental group (while in topological context, people usually do not restrict to finite covers).

One possible answer is that in the case of varieties over $\mathbb{C}$, infinite-degree topological covers do not have to be algebraic (as can be seen by considering the exponential map to affine line).

Remark 5.3. In fact, under additional hypotheses we can show that étale morphisms *have to* be quasi-finite so infinite-degree topological covers can not be algebraic. The following argument is due to Jason Starr.

Lemma. Let $i : X \to Z$ be a separated morphism between irreducible schemes. If there exists a covering of X by open affines U such that each restriction $i|_U$ is an open immersion, then i is an open immersion.

Proof. Possibly after replacing Z with the open image of i we may assume that i is surjective. Then we have to prove that i is an isomorphism. The inverse isomorphism $i^{-1} : Z \to X$ can be constructed by gluing. Let U and V be nonempty open subschemes of X. We have to verify the cocycle condition for i^{-1}, i.e. that $i^{-1}(i(U) \cap i(V))$ equals $U \cap V$.

Let Y^o be a nonempty open affine subset of the open intersection $i(U) \cap i(V)$. Denote by X^o the inverse image $i^{-1}(Y^o)$. Since X is irreducible, the intersections of nonempty open subsets $U \cap X^o$ and $V \cap X^o$ are dense. Denote these by U^o and V^o. By construction, each of the following restrictions of i is an isomorphism,

$$i_U : U^o \to Y^o, \quad i_V : V^o \to Y^o.$$

These isomorphisms agree on $U^o \cap V^o = (U \cap V) \cap X^o$.

Since i is separated and since Y^o is affine, the scheme X^o is separated. Define j to be the automorphism of X^o whose restriction to U^o equals $i_V^{-1} \circ i_U$ and whose restriction to V^o equals $i_U^{-1} \circ i_V$. They can be glued since i_U and i_V agree on $U^o \circ V^o$. Moreover, j equals the identity on $U^o \circ V^o$. Since j and the identity agree on the dense open $U^o \circ V^o$, and since X^o is separated, the morphism j equals the identity. Thus, U^o equals V^o. Since we can cover $i(U) \cap i(V)$ by such open affines, it follows that $i^{-1}(i(U) \cap i(V))$ equals $U \cap V$. □

Proposition. A separated surjective étale morphism from a connected scheme to a Noetherian, separated, excellent, integral, and normal scheme is quasi-finite.

Proof. Let $f : X \to Y$ be the morphism under consideration. By definition, f is locally quasi-finite so we have to verify that it is quasi-compact.

Since Y is normal and f is étale, we get that X is normal. As X is connected, it is irreducible and has a unique generic point η.

Since Y is excellent, the "integral closure" of Y in the "function field" $\kappa(\eta)$ is a finite, strongly dominant morphism whose domain is normal,

$$g : Z \to Y.$$

By the universal property of the normalization, there exists a unique morphism of schemes compatible with the specified morphisms to Y,

$$i : X \to Z.$$

By Zariski's Main Theorem, working locally on X with opens that are quasi-compact over Y, the morphism i is an locally an open immersion.

By the lemma, the morphism i is an open immersion. Since Z is a scheme finite over a Noetherian scheme, it is a Noetherian scheme. As X is an open subset of Z, it is quasi-compact. □

Note that under hypotheses of the proposition, f is finite iff the cardinality of the fibers over closed points is constant, see EGA IV, Corollary 18.2.9.

It should be noted that in SGA3 X6, a different version of étale fundamental group was introduced, its defining property being that $Hom(^{SGA3}_1(X, x), \Gamma)$ is in bijection with Γ-torsors trivialized at x, for any discrete group Γ. Étale fundamental group as defined above is the profinite completion of $\pi_1^{SGA3}(X, x)$.

Another version of étale fundamental group has been considered in [3], the so-called pro-étale fundamental group. For a connected scheme X whose underlying topological space is locally Noetherian, it parameterizes the schemes $Y \to X$ that are ÃĪtale and satisfy the valuative criterion of properness. Since Y is not necessarily of finite type over X, the map does not have to be proper so we are not restricted to finite covers.

The pro-étale fundamental group generally speaking constains more information than $\pi_1^{SGA3}(X, x)$. The group $\pi_1^{\mathrm{SGA3}}(X, x)$ is the pro-discrete completion of $\pi_1^{\text{proét}}(X, x)$.

In the case of a geometrically unibranch scheme, π_1, π_1^{SGA3} and $\pi_1^{\text{proét}}$ all match. For the projective line over an algebraically closed field with two points identified one can show that $\pi_1^{\text{proét}}(X, x) \approx \pi_1^{\mathrm{SGA3}}(X, x) \approx \mathbb{Z}$. Already in the case of an elliptic curve with two points identified, $\pi_1^{\text{proét}}(X)$ gives more information than $\pi_1^{\mathrm{SGA3}}(X)$.

6. Étale homotopy type

In this section we explain how to associate étale homotopy type to a locally Noetherian scheme.

Let X be a scheme. A simplicial object $U.$ in the small étale site of X is called a hypercovering if

- $U_0 \to X$ is a surjective étale morphism;
- for every n the natural map $U_{n+1} \to cosk_n(U.)_{n+1}$ is a surjective étale morphism (here $cosk_n$ refers to n-coskeleton).

Étale hypercoverings form a category; its homotopy category $HC(X_{et})$ is the category that has the same objects and where homotopic maps (the notion of homotopy being the classical one from the theory of simplicial objects) are identified.

Now assume that X is locally Noetherian. Then every scheme Y étale over X is a finite disjoint union of connected schemes. Write $\pi_0(Y)$ for the set of connected components of Y . Denote by $\pi_0(U.)$ be the simplicial set obtained from the simplicial étale hypercovering $U.$ by applying π_0. Since $HC(X_{et})$ is cofiltering, the functor $\pi_0 : HC(X_{et}) \to HC(SimpSet)$ defines an object of the pro-category of the homotopy category of simplicial sets. By considering the topological realisation of this pro-object, we get an object $Et(X)$ in the pro-category of the homotopy category of CW-complexes. This object is called the étale homotopy type of X.

Remark 6.1. This construction is manifestly motivated by Čech nerve theorem from algebraic topology. The reason we need to take the direct limits is that typically one can not find covers by contractible open sets in étale topology (in fact, over an algebraically closed field of positive characteristic k, the only smooth variety that is étale 2-connected is $\mathrm{Spec}(k)$ [4]).

The need to take direct limits is responsible for the fact that we only get étale homotopy type rather than étale topological type (the category of étale hypercoverings of a given scheme, generally speaking, is not cofiltering while its homotopy category is). There are certain functorial constructions of étale topological type (see e.g. [5]) but we will not review them here.

Remark 6.2. For an approach that applies to schemes that are not locally Noetherian, see [11].

It is fairly easy to show that for any abelian group A, the cohomology of $Et(X)$ with coefficients in A coincides with the sheaf cohomology of the constant sheaf on étale site associated to A. If one picks a geometric point in X, it is possible to define pointed étale homotopy type, and its fundamental group will coincide with étale fundamental group as defined in the previous section. We see therefore that our construction of étale homotopy type is not entirely unreasonable.

For a pointed connected geometrically unibranch scheme of finite type over over $\mathrm{Spec}\,\mathbb{C}$, $Et(X)$ is canonically identified with $\widehat{X(\mathbb{C})}$, the profinite completion of the homotopy type of the set of complex points of X endowed with analytic topology. If the scheme in question is not geometrically unibranch the étale homotopy type is not necessarily profinite complete but its profinite completion agrees with $\widehat{X(\mathbb{C})}$.

Remark 6.3. For smooth varieties, the identification mentioned above can be seen fairly directly at the level of cohomology. Assume X is a smooth complex variety. Artin [10] has established that any point in X has a Zariski open neighbourhood which is a total space of an iterated fibration with fiber being an affine curve. Such a neighbourhood has the following remarkable properties:

(1) for a locally constant constructible abelian étale sheaf F and a geometric point $x \in U$, the natural map $H^*(\pi_1^{et}(U,x), F_x) \longrightarrow H^*_{et}(U,F)$ is an isomorphism;
(2) the fundamental group of the set of complex points $\pi_1(U(\mathbb{C}),x)$ is an iterated extension of free groups by free groups;
(3) higher homotopy groups of $U(\mathbb{C})$ vanish.

Therefore, we get a chain of isomorphisms

$$H^*_{et}(U,F) \cong H^*(\pi_1^{et}(U,x), F_x) \cong H^*(\pi_1(U(\mathbb{C}),x), F_x) \cong H^*(U(\mathbb{C}),F).$$

where the first isomorphism comes from property (1), the second comes from property (2) and the fact that $\pi_1^{et}(U,x)$ is the profinite completion of $\pi_1(U(\mathbb{C}),x)$, and the third comes from property 3 and the fact that complex varieties are CW complexes. This resulting isomorphism $H^*_{et}(U,F) \cong H^*(U(\mathbb{C}),F)$ can be "bootstrapped" to X using hypercovering spectral sequence.

It should be noted that for schemes over Spec $\mathbb{C}$ étale homotopy type, generally speaking, does not determine the homotopy type of the associated complex-analytic space. For some explicit examples, see [6].

The amount of topological information packaged in étale homotopy type can be illustrated by Sullivan's conjecture. Recall that Sullivan's conjecture states that for a prime number p and a finite $\mathbb{Z}/p$-CW-complex X, the natural map $F_{\infty p}(X^{\mathbb{Z}/p}) \to (F_{\infty p}X)^{h\mathbb{Z}/p}$ is a homotopy equivalence (here $F_{\infty p}$ stands for Bousfield–Kan p-completion). So for example, if $p = 2$ and X is the CW-complex associated to a set of complex points of a normal projective variety defined over real numbers (with $\mathbb{Z}/2$-action being the complex conjugation) then $X^{\mathbb{Z}/2}$ is identified with the set of real points of the variety and the conjecture states that the Bousfield–Kan 2-completion of the latter can be recovered from the Bousfield–Kan 2-completion of the set of complex points, which can be recovered from étale homotopy type; invariants like the $\mathbf{F}_2$-cohomology of $X(\mathbf{R})$ can be recovered this way.

Let us conclude this section with some examples for which some explicit information about étale homotopy type is known.

- Let k be a field with finite absolute Galois group. Then Spec k_{sep} is an étale contractible cover of Spec k so étale homotopy type of Spec k is weakly equivalent to the Čech nerve of Spec $k_{sep} \to$ Spec k. The latter can be easily seen to coincide with the bar construction of BG for $G = Gal(k_{sep}/k)$. Arguably, this example is somewhat boring since by Artin–Schreier theorem up to weak equivalence, the only étale homotopy type we get is $B\mathbb{Z}/2$;
- Every connected affine Noetherian geometrically unibranch scheme has vanishing higher étale homotopy groups [7];
- étale homotopy type of Spec $\mathbb{Z}$ is contractible, see e.g. [8].

7. Applications

As the reader may have already guesses, the machinery of étale cohomology can be used to establish Weil conjectures. More precisely, from a Lefschetz-type trace formula one gets (1), from Poincaré duality one gets (2) and from the comparison between étale topology and complex-analytic topology one gets (4). The third hypothesis, the most difficult one, does not immediately follow from some formal properties of étale cohomology and requires additional ideas.

Étale cohomology finds many other applications in number theory. The way it is typically used is the following: for any scheme X over a field k and an abelian sheaf on X, the groups $H^i_{et}(X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\ k_{sep}, F_{k_{sep}})$ admit a canonical action of $Gal(k_{sep}/k)$, i.e. étale cohomology groups furnish Galois representations. For example, the cohomology of the moduli stack of shtukas can be used to implement Langlands correspondence in the function field case. On the other hand, sometimes Galois representation we get can be used to understand better the algebraic geometry of the original scheme (see Neron–Ogg–Shafarevich criterion).

Étale homotopy theory also has some applications in topology and geometry. For example, it can be used to prove the following theorem due to D. Cox.

Theorem 7.1. *A connected real algebraic variety X of dimension n has a real point iff $H^i_{et}(X, \mathbb{Z}/2) \neq 0$ for some $i > 2n$.*

It was also used by Friedlander to give a proof of Adams connjecture.

References

[1] Quasi-projectivity of normal varieties. International Mathematics Research Notices, Volume 2013, Issue 17, 1 January 2013, Pages 3878-3885

[2] Schröer, S. (2017). Geometry on totally separably closed schemes. Algebra & Number Theory, 11(3), 537-582.

[3] The pro-étale topology for schemes. (English, French summary) Astérisque No. 369 (2015), 99-201.

[4] Étale contractible varieties in positive characteristic. Algebra Number Theory Volume 8, Number 4 (2014), 1037-1044.

[5] Étale Homotopy of Simplicial Schemes. (AM-104), Volume 104

[6] Jean-Pierre Serre, Exemples de variétés projectives conjuguées non homéomorphes, C. R. Acad. Sci. Paris 258 (1964), 4194-4196 (French).

[7] Achinger, P. Wild ramification and $K(\pi, 1)$ spaces. Invent. math. (2017) 210: 453.

[8] T. Schlank, MathOverflow answer

[9] T. Poonen. MathOverflow answer

[10] SGA 4, Exposé XI, Comparaison avec la cohomologie classique : cas d'un schéma lisse, par M. Artin

[11] Carchedi, D. On the étale homotopy type of higher stacks, arXiv preprint