A walk through the world of manifolds

A. Kazhymurat

November 2018

1 Conventions

In this note, we consider some geometric structures on manifolds relevant to complex geometry. We only state the definitions and review some of the fundamental properties. Naturally, there are a lot of omissions, e.g. we do not explicitly mention holonomy or Berger's list and we do not describe the classification of closed complex surfaces or the symplectic geography problem.

A convention is that a topological manifold is a second countable locally Euclidean Hausdorff space.

2 Smooth structure

Let M be a closed topological manifold of dimension n. If

- $n \leq 3$, then M admits a unique (up to diffeomorphism) smooth structure.
- $n \geq 5$, then M admits only finitely many (possibly 0) pairwise non-diffeomorphic smooth structures.

In dimension $n = 4$, there exist closed topological manifolds that admit countably many distinct smooth structures (e.g. the underlying topological manifold of a quartic hypersurface in $\mathbb{C}P^3$) as well as closed topological manifolds not admitting a smooth structure (e.g. E8 manifold).

3 Riemannian structure

Definition 1. *A Riemannian metric on a smooth manifold M is a smooth tensor field of type (2, 0) such that for all $x \in M$ and for any non-zero $u \in T_xM$, we have $g(u, u) > 0$.*

Via a partition of unity argument, one can see that every smooth manifold admits a Riemannian metric.

Definition 2. *An affine connection on a smooth manifold M is a bilinear map*

$$\Gamma(M, TM) \times \Gamma(M, TM) \to \Gamma(M, TM), \qquad (X, Y) \to \nabla_X Y$$

such that

- *for any smooth function f on M, we have*

$$\nabla_{fX}Y = f\nabla_X Y.$$

- *for any smooth function f on M, we have*

$$\nabla_X(fY) = df(X)Y + f\nabla_X Y.$$

If we have a smooth interval $c : [0,1] \to M$ then for any vector $v \in T_{c(0)}M$ there exists a vector field X such that

$$\nabla_{c'(t)}X = 0, \qquad X(c(0)) = v.$$

The vector $v' = X(c(1))$ is called the parallel transport of v along c.

Definition 3. *An affine connection ∇ on a Riemannian manifold (M,g) is called Levi–Civita connection if*

- $\nabla_X Y - \nabla_Y X = [X,Y]$
- *for any point $p \in M$ and v, $u \in T_pM$ and for any smooth $c : [0,1] \to M$ we have $g(u,v) = g(u',v')$ where u' is the parallel transport of u and v' is the parallel transport of v.*

Theorem 1. *For any Riemannian manifold (M,g), there exists a unique Levi–Civita connection.*

Proof. On any local chart $U \subset M$ with nowhere vanishing linearly independent vector fields $\partial_1, \ldots, \partial_n$ we have that Levi–Civita connection has to satisfy the following identity

$$\langle \nabla_{\partial_i}\partial_j, \partial_k\rangle = \tfrac{1}{2}\left(\partial_i g_{jk} - \partial_k g_{ij} + \partial_j g_{ik}\right).$$

This means that LV connection is uniquely defined if it exists. For existence, verify that the assignment defined by the following formula is a Levi-Civita connection.

$$g(\nabla_X Y, Z) = \tfrac{1}{2}\Big\{X\big(g(Y,Z)\big)+Y\big(g(Z,X)\big)-Z\big(g(X,Y)\big)+g\big([X,Y],Z\big)-g\big([Y,Z],X\big)-g\big([X,Z],Y\big)\Big\}.$$

□

Definition 4. *The Riemann curvature tensor of a Riemannian manifold (M,g) with Levi–Civita connection ∇ is the tensor R defined by the following expression*

$$R(u,v)w = \nabla_u\nabla_v w - \nabla_v\nabla_u w - \nabla_{[u,v]}w$$

The information of Riemann tensor can be re-packaged into a slightly different object, sectional curvature.

Definition 5. *Let (M, g) be a Riemannian manifold with Riemann tensor R. Then sectional curvature of (M, g) is a function on the Grassmann bundle of tangent 2-planes of M*

$$K(\sigma) = \frac{\langle R(u, v)v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}$$

where u and v are elements of some basis of $\sigma \subset T_pM$.

It is easy to see that sectional curvature is well-defined (i.e. does not depend on the choice of a basis of tangent 2-plane).

Theorem 2. *Let (M, g) be a Riemannian manifold with Riemann tensor R and sectional curvature k. Then the following formula holds*

$$6g(R(x, y)z, w) = k(x + w, y + z) - k(x, y + z) - k(w, y + z)$$

$$-k(y + w, x + z) + k(y, x + z) + k(w, x + z)$$

$$-k(x + w, y) + k(x, y) + k(w, y)$$

$$-k(x + w, z) + k(x, z) + k(w, z)$$

$$+k(y + w, x) - k(y, x) - k(w, x)$$

$$+k(y + w, z) - k(y, z) - k(w, z).$$

Definition 6. *The Ricci curvature tensor of a Riemannian manifold (M, g) with The Riemann curvature tensor R is the tensor* Ric *which in a local chart with coordinates $x_1, \ldots, x_n$ is defined by the following expression*

$$\mathrm{Ric} = R_{ij}dx^i \otimes dx^j, \qquad R_{ij} = R^k_{ikj}, \qquad R^i{}_{jkl} = dx^i(R(\partial_k, \partial_l)\partial_j).$$

A Riemannian metric g is said to have positive (resp. negative) Ricci curvature if for any point $p\ inM$ and any non-zero vector $u \in T_pM$ the number $\mathrm{Ric}(u, u)$ is positive (resp. negative).

Definition 7. *The scalar curvature of a Riemannian manifold (M, g) with Ricci curvature* Ric *is a function s defined as follows*

$$s = \mathrm{tr}_g\mathrm{Ric}.$$

In local coordinates $x_1, \ldots, x_n$, if $\mathrm{Ric} = R_{ij}dx^i \otimes dx^j$ then $s = g^{ij}R_{ij}$. Simple linear algebra shows that at any point $p \in M$, we can pick local coordinates $x_1, \ldots, x_n$ so that $g = \sum_{i=1}^n dx^i \otimes dx^i$. This means, among other things, that if we have an orthonormal basis $e_1, \ldots, e_n$ of T_pM, then $s(p) = \sum_{i=1}^n \mathrm{Ric}(e_i, e_i)$.

3.1 Metrics of special curvature

It is fairly natural to consider Riemannian metrics which have constant curvature, in some sense. So far, we have introduced 3 inequivalent notions of curvature: sectional curvature, Ricci curvature and scalar curvature. In dimension 2, 3 notions of curvature are equivalent and in dimension 3, the first two are equivalent. However, in higher dimensions all 3 notions of curvature diverge.

- A Riemannian manifold of dimension n that has constant sectional curvature has to be locally isometric (after a suitable rescaling of the metric) to one of the following: S^n, n-dimensional sphere with round metric, $\mathbb{R}^n$, Euclidean space with flat metric, or H^n, hyperbolic space endowed with the metric of constant scalar curvature. Therefore, the geometry of spaces of constant sectional curvature is fairly limited.
- Any Riemannian metric g on a closed manifold can be conformally rescaled to a metric $g' = e^{2f}g$ ($f \in C^\infty(M)$) that has constant scalar curvature. Therefore, the geometry of spaces of constant scalar curvature is very broad.

The study of Riemannian metrics of constant Ricci curvature, i.e. metrics satisfying

$$\mathrm{Ric} = \lambda g, \qquad \lambda \in \mathbb{R},$$

also known as Einstein metrics is very interesting. The existence of such metrics imposes (at least in some dimensions) topological restrictions on the underlying manifold. It also can be shown that the moduli space of Einstein metrics on a closed manifold is finite-dimensional, in some sense.

The existence of an Einstein metric on a closed oriented 4-manifold M implies that

$$\chi(M) \geq \frac{3}{2}|\tau(M)|$$

where $\chi(M)$ is the Euler characteristic of M and $\tau(M)$ is the signature of M (Hitchin–Thorpe inequality). This, in some sense, shows that Einstein metrics are very special (the existence of constant scalar curvature metric on a closed manifold, for example, does not impose any topological restrictions).

Author is not aware of topological obstructions to the existence of Einstein metrics in dimension ≥ 5.

3.2 Sign of curvature and topology

The following theorem due to Gauss-Bonnet shows how Riemannian metric interacts with topology in dimension 2.

Theorem 3. *If M is a closed manifold of dimension 2, then for any Riemannian metric g on M we have*

$$\int_M s_g dA_g = 4\pi\chi(M)$$

where $s(g)$ is the scalar curvature of g, dA_g is the area element and $\chi(M)$ is the Euler characteristic of M.

In particular, for surfaces, the existence of a Riemannian metric with negative Ricci curvature imposes topological restrictions. This is no longer true in higher dimensions—by a result of Lohkamp, every smooth manifold of dimension $n \geq 3$ admits a Riemannian metric of negative Ricci curvature.

However, the existence of metrics with positive Ricci curvature still carries non-trivial topological information in higher dimensions (for example, by a theorem of Myers a closed manifold admitting such a metric has to have finite fundamental group).

4 Almost complex structure

If M is a smooth manifold, then we can consider almost complex structures on it.

Definition 8. *An almost complex structure (a.c.s.) on M is a smooth global section of $End(TM)$ satisfying $J^2 = -Id$.*

Assume that M admits an almost complex structure J. By considering the determinant of the linear transformation of the fiber of TM over some point $x \in M$ induced by J, one sees that M has to be even-dimensional.

Any almost complex manifold is canonically oriented: by picking Riemannian metric on M satisfying $g(\cdot,\cdot) = g(J\cdot, J\cdot)$ and considering the top power of the non-degenerate 2-form $\omega = g(J\cdot,\cdot)$, we endow M with an orientation. By linear algebra, at any point $p \in M$ we can find a basis of T_pM of the form $e_1, Je_1, \ldots, e_n, Je_n$, so the orientation we produced does not depend on the choice of the metric.

In real dimension 2, all orientable manifolds admit an almost complex structure. This is no longer true starting from dimension 4.

Theorem 4. *A 4-manifold M admits an almost-complex structure J if and only if there exists $h \in H^2(M,\mathbb{Z})$ such that*

$$h^2 = 3\sigma(X) + 2\chi(X) \quad \text{and} \quad h \equiv w_2(X) \mod 2$$

where $\chi(M)$ is the Euler characteristic and $\sigma(M)$ is the signature of M.

This shows, for example, that 4-dimensional sphere S^4 can not admit an almost complex structure.

In real dimension 6, we have the following result due to Wall.

Theorem 5. *Let M be a smooth oriented 6-manifold. Then M has an almost complex structure iff $w_3(M) = 0$. When this is so, there is only one homotopy class of almost complex structure for each $c_1 \in H^2(M,\mathbb{Z})$ whose mod 2 reduction is $w_2(M)$.*

Remark 1. *In real dimension 4, the classification of almost complex structures up to homotopy is not so simple. For example, on K3 surface there exists a homotopy class of non-integrable almost complex structures with null* c_1 *[3].*

It has been proven [1] that if a rational homology sphere of dimension n admits an almost complex structure then $n = 2, 6$. For $n = 2$, an example of an almost complex manifold is Riemann sphere. For $n = 6$, the space (diffeomorphic to S^6) of unit purely imaginary octonions provides an example of an almost complex manifold.

4.1 Chern classes

It can be shown that there is a unique way to assign to every complex vector bundle E over a topological space X a characteristic class $c(E) \in H^*(X, \mathbb{Z})$ satisfying the following axioms:

- the projection of $c(E)$ to H^0 is 1 for all E;
- c commutes with pullbacks under continuous mappings:
- for two complex vector bundles E and F over a given space $c(E \oplus F) = c(E) \cup c(F)$.
- for all $n > 0$, if $O(1)$ denotes the vector bundle of homogeneous polynomials of degree 1 over $\mathbb{C}P^n$, we have $c(O(1)) = 1 + H$, where $H \in H^2(\mathbb{C}P^n, \mathbb{Z})$ is the class dual to hyperplane.

Since the tangent bundle of an almost complex manifold (M, J) is naturally a complex vector bundle, we may consider the Chern class of TM. When the almost complex structure is clear from the context, $c(TM)$ is referred to as the Chern class of M. Its projection to $H^{2k}(M, \mathbb{Z})$ is called k-th Chern class and is denoted by $c_k(M)$. On a closed manifold, the pairing of the fundamental class and any product of Chern classes having top rank is called a Chern number (so for a closed almost complex surface we have $\int_M c_1^2$ and $\int_M c_2$).

5 Complex structure

Let M be a smooth manifold.

Definition 9. *A complex manifold is a smooth manifold with an atlas with biholomorphic transition maps.*

Note that a complex manifold has a canonical almost complex structure (locally, an almost complex structure is obtained by pulling it back from an open ball in $\mathbb{C}^n$; this construction gives a globally well-defined almost complex structure since the transition maps are holomorphic). Moreover, a complex manifold is determined, up to biholomorphism, by its associated almost complex structure. This follows from the existence of local holomorphic coordinates, i.e. coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ such that

$$J\frac{\partial}{\partial x_i} = \frac{\partial}{\partial y_i}, \qquad J\frac{\partial}{\partial y_i} = -\frac{\partial}{\partial x_i}.$$

Definition 10. *An almost complex structure J on a smooth manifold M is called integrable if M can be given a structure of complex manifold so that the associated almost complex structure is J.*

The following is a holomorphic version of Frobenius theorem. For the proof, see [2]. Recall that a holomorphic bundle E of rank k on a complex manifold X of dimension n is called integrable if X has an open covering U_i and there exist holomorphic submersions $\phi_i : U_i \to \mathbb{C}^{n-k}$ such that $\ker d\phi_i = E|_{U_i}$.

Theorem 6. *Let X be a complex manifold of dimension n, and let E be a holomorphic vector subbundle of rank k of the holomorphic tangent bundle TX . Then E is integrable if and only if we have $[E, E] \subset E$.*

The following is a theorem due to Newlander–Nirenberg.

Theorem 7. *An almost complex structure J is integrable if and only if*

$$[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] = 0$$

for all smooth vector fields X, Y.

Sketch of a proof in the case the manifold and almost complex structure are real-analytic. The 'only if' part is trivial. A statement is local so we can assume that there is a diffeomorphism $f : X \to U \subset \mathbb{R}^{2n}$ from X to an open ball. Since J is real-analytic, possibly after restricting U, we may assume that it is given by convergent power series; so we can extend it to an almost complex structure on an open neighbourhood U' of U in $\mathbb{C}^{2n}$.

If we consider $-i$-eigenspaces of I, we get a holomorphic distribution E on U'. The condition in the theorem implies that $[E, E] \subset E$. From the previous theorem, possibly after restricting U', we get a holomorphic submersion $U' \to \mathbb{C}^n$ whose restriction to U is a local diffeomorphism. It can be shown that the pullback of the standard almost complex structure to U from $\mathbb{C}^n$ coincides with I. □

Remark 2. *In the case we assume smoothness, rather than real-analyticity, the proof of the theorem involves some non-trivial analysis. The uprgrade in regularity one gets is similar to the fact that C^1 solutions of Cauchy–Riemann equations have to be real-analytic.*

5.1 Low dimension

Every oriented surface S admits a complex structure. To see this, pick a Riemannian metric g on S. It is known that around any point, there exist local coordinates u, v such that

$$g = e^{2f}(du^2 + dv^2)$$

for some smooth function f. The proof of this statement typically involves some analysis (see [4]). Since S is oriented, charts can be chosen so that (du, dv) is an oriented basis of T_pS at every point. Then we can define an almost complex structure by sending du to dv. It is easy to see that this almost complex structure has to be integrable (note that Nijenhuis tensor is skew-symmetric and that the diagonal component is zero). Therefore, we get a bona-fide complex structure.

In complex dimension 2, one has a classification of closed complex manifolds. We will not consider it here.

Our understanding of complex manifolds is pretty bad starting from complex dimension 3. Their topology can be very diverse; for example, any finitely presented group is the fundamental group of a complex 3-fold (however, this is not true in real dimension 4; free groups of rank $r > 2$ cannot be realized as fundamental groups of closed complex surfaces [5]).

5.2 Hodge numbers

Let X be a complex manifold of dimension n. Denote by $\Omega^{1,0}$ the vector bundle of holomorphic 1-forms on X and by $\Omega^{0,1}$ the vector bundle of antiholomorphic 1-forms on X. We define $\Omega^{p,q} = \wedge^p\Omega^{1,0} \wedge \wedge^q\Omega^{0,1}$. It is easy to see that $d(\Omega^{p,q}) \subset \oplus_{r+s=p+q+1}\Omega^{r,s}$. By projecting to the appropriate summand, we define Dolbeault operators $\partial : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\bar{\partial} : \Omega^{p,q} :\to \Omega^{p,q+1}$. It is easy to see that $\partial^2 = \bar{\partial}^2 = 0$ and $d = \partial + \bar{\partial}$.

This means that we have a double complex with the (p, q)-term being $\Gamma(X, \Omega^{p,q})$, horizontal differential being ∂ and vertical differential being $\bar{\partial}$. If X is closed, the spectral sequence associated to the double complex will converge to the cohomology of the total complex (which is identified with the complexified de Rham complex)

$$E_1^{p,q} = \left\{\alpha \in \Gamma(X, \Omega^{p,q} | \bar{\partial}\alpha = 0\right\} / \bar{\partial}\Gamma(X, \Omega^{p,q-1}) \to H^{p+q}(X, \mathbb{C}).$$

This spectral sequence is referred to as the Frölicher spectral sequence. It always degenerates at the first page for closed complex curves and surfaces. However, for any given n, there exists a closed complex manifold such that the Frölicher spectral sequence does not degenerate at E_n [6].

Dolbeault theorem says that for a closed complex manifold $E_1^{p,q}$ is canonically isomorphic to $H^q(X, \Omega^{p,0})$. The dimension of $H^q(X, \Omega^{p,0})$, denoted $h^{p,q}(X)$, is called the (p, q)-th Hodge number of X.

Serre duality states that for a closed complex manifold X and a holomorphic vector bundle V we have perfect pairing

$$H^q(X, V) \times H^{n-q}(X, V^* \otimes \Omega^{n,0}) \to H^n(X, \Omega^{n,0}) \approx \mathbb{C}.$$

In particular, $h^{p,q} = h^{n-q,n-p}$ for any closed complex manifold.

5.3 Moishezon manifolds

Definition 11. *Let X be a connected complex manifold with the sheaf of holomorphic functions O_X. The sheaf of meromorphic functions K_X is defined as* *the sheafification of the presheaf sending an open set to the ring of fractions of $O_X(U)$.*

Definition 12. *The field of meromorphic functions of X is the ring of global sections of K_X.*

It is easy to see that the field of meromorphic functions is indeed a field.

Definition 13. *The transcendence degree of the field of meromorphic functions on a connected closed complex manifold X is called the algebraic dimension of X.*

The following is a theorem due to Siegel–Remmert–Thimm.

Theorem 8. *The algebraic dimension of connected closed complex manifold does not exceed its complex dimension.*

In the class of complex tori of complex dimension $n \geq 2$, one can find manifolds of algebraic dimension a for any $0 \leq a \leq n$. For example, let us consider a complex torus $X = \mathbb{C}^2/(\tau, \mathrm{Id}_2)\mathbb{Z}^4$ with τ being a 2x2 matrix with invertible imaginary part. The complex structure J on X can be written as follows

$$J = \begin{bmatrix} y^{-1}x & y^{-1} \\ -y - xy^{-1}x & -xy^{-1} \end{bmatrix}$$

$$x = \mathrm{Re}\,\tau, \qquad y = \mathrm{Im}\,\tau.$$

It can be shown [7] that the algebraic dimension X is equal to the following

$$a(x) = \frac{1}{2}\max\{\mathrm{rank}\ J^{\intercal}E | E \in NS(X) \text{ with } J^{\intercal}E \geq 0\}$$

where $NS(X)$ is the set of Hermitian forms (considered as square matrices) $H : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}$ with $\mathrm{Im} H((\tau, \mathrm{Id}_2)\mathbb{Z}^4 \times (\tau, \mathrm{Id}_2)\mathbb{Z}^4) \subset \mathbb{Z}$. Now we can see that any value of algebraic dimension $0 \leq a(X) \leq 2$ is possible:

- Take $\tau = i\mathrm{Id}_2$ for $a(X) = 2$;
- Take
$$\tau = \begin{bmatrix} 0 & \sqrt{2} - i\sqrt{3} \\ 0 & 0 \end{bmatrix}$$
for $a(X) = 1$;
- Take
$$\tau = \begin{bmatrix} i2^{\frac{1}{3}} & \sqrt{3} \\ -\sqrt{5} & i \end{bmatrix}$$
for $a(X) = 0$.

Remark 3. *The almost complex structure on the six-sphere that we produced above is not integrable (if we consider S^6 as the space of unit purely imaginary octonions then Nijenhuis tensor is proportional to the associator in oction algebra). It is not known if S^6 admits an integrable almost complex structure (but it is known that there is no integrable a.c.s. J satisfying $g(\cdot,\cdot) = g(J\cdot, J\cdot)$ with respect to the round metric g [8]). It has been shown that such a complex structure would not admit non-constant meromorphic functions (e.g. $a(S^6) = 0$) [9] and that either its $h^{1,1}$ or $h^{2,0}$ has to be positive [10]. It also can be shown that existence of a complex structure on S^6 would imply the existence of an exotic complex structure on $\mathbb{C}P^3$. [11].*

Definition 14. *A closed connected complex manifold is called Moishezon if its dimension equals its algebraic dimension.*

It can be shown that a closed Moishezon manifold has positive second Betti number (see, for example, [9]).

6 Symplectic structure

Arnol'd conjecture (symplectic structure on the cotangent bundle sees exotic smooth structures on the base)

Let M be a smooth manifold.

Definition 15. *A symplectic structure on M is a closed nowhere degenerate 2-form on M.*

The existence of symplectic structure imposes restrictions on the underlying manifold. For example, a closed manifold admitting a symplectic structure

- admits an almost complex structure (to see this, make a fiberwise calculation without picking a basis);
- has non-zero Betti numbers b_{2k} for $1 \leq k \leq n$ (consider the classes $[\omega^k]$).

The following statement is frequently referred to as Moser's lemma.

Theorem 9. *If two symplectic forms can be joined by a path in the space of symplectic forms with fixed cohomology class, then the symplectic manifolds (M, ω_1) and (M, ω_2) are symplectomorphic (i.e. there is a diffeomorphism mapping one form to another).*

Remark 4. *Note that there do exist non-symplectomorphic symplectic forms with $[\omega_1] = [\omega_2]$ joined by a path of symplectic forms (though this does not happen on open manifolds by a version of h-principle). It is essential that cohomology class is the same at all points on the path, not only for the end points. For a detailed discussion see [27].*

An almost complex structure J on M is said to be compatible with a symplectic form ω if for any $x \in M$ we have $\omega(v, Jv) > 0$ for non-zero $v \in T_xM$ and

$\omega(u, v) = \omega(Ju, Jv)$ for any $u,\ v \in T_xM$. It is known that for any symplectic form ω, the space of compatible almost complex structures is non-empty and contractible (in compact-open C^∞-topology). In particular, to any symplectic manifold (M, ω), we can associate Chern class $c(TM) \in H^*(M, \mathbb{Z})$.

The topology of symplectic manifolds is truly diverse, as evidenced by the following result due to Gompf.

Theorem 10. *Any finitely presented group is a fundamental group of some closed symplectic manifold of dimension 4.*

7 Kähler structure

Definition 16. *A Kähler manifold is a smooth manifold M endowed with a symplectic structure ω and a compatible integrable almost complex structure J.*

Note that by the definition of compatible a.c.s., a Kähler manifold (M, ω, J) has a Riemannian metric $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$.

Let (M, ω, J) be a Kähler manifold. It is easy to see that $N \subset M$ is a complex submanifold, then $\omega|_N$ defines a symplectic form on N (in particular, N is canonically a Kähler manifold). Note that the converse is not always true (i.e. a symplectic submanifold does not have to be complex).

Remark 5. *It is conjectured that for every closed smooth symplectic submanifold $\Sigma \subset \mathbb{C}P^2$ there exists a family of closed smooth symplectic submanifolds $\Sigma_t \subset \mathbb{C}P^2$ ($0 \leq t \leq 1$) with $\Sigma_0 = \Sigma$ and Σ_1 a complex submanifold of $\mathbb{C}P^2$. This has been established in degree ≤ 17 by Siebert–Tian.*

7.1 Topological properties of Kähler manifolds

If M is a closed Kähler manifold then

- Lie algebra sl_2 acts non-trivially on $H^*(M, \mathbb{C})$.
- Frölicher spectral sequence of the underlying complex manifold degenerates at E_1
$$H^k(M, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}(M).$$
- Hodge symmetry holds, i.e. $H^{p,q}(M) = \overline{H^{q,p}(M)} \subset H^{p+q}(M, \mathbb{C})$. This combined with the previous point means that odd Betti numbers are even.
- The fundamental group $\pi_1(M)$ has to satisfy a number of restrictions, collectively referred to as Kählerness (compare with the theorems of Gopmf and Taubes mentioned above).
- Rational homotopy type is formal. Symplectic nilmanifolds other than torus are not formal so they can not be Kähler.

- The cup product with $[\omega]^i$ induces an isomorphism $H^{n-i}(M, \mathbb{R}) \to H^{n+i}(M, \mathbb{R})$ (the property known as Hard Lefschetz theorem). This never holds for symplectic nilmanifolds other than torus [15].

Remark 6. *It was originally shown that closed Kähler manifolds have formal homotopy type over $\mathbb{R}$ [12]. However, later result of Sullivan [13] has shown that formality over any field of characteristic 0 implies formality over $\mathbb{Q}$.*

Still, these propeties do not characterize Kähler manifolds completely. There exist closed smooth manifolds of real dimension 6 which admit both complex and symplectic structure and satisfy all of the properties above but they do not admit Kähler structure [14].

Remark 7. *Any Moishezon manifold is bimeromorphic to a Kähler manifold. Therefore, some of the topological properties of Kähler manifolds are shared by Moishezon manifolds. For example, Hodge decomposition also exists for closed Moizheson manifold and fundamental groups of closed Moishezon manifolds are Kähler. However, there do exist closed Moishezon manifolds not homotopy equivalent to a closed Kähler manifold (see [16] for an example of a Moishezon 3-fold such that the top power of any closed 2-form is exact).*

On some manifolds, Kähler structure is uniquely determined by the topology. This is the case for complex projective spaces (see, for example, [17]).

Theorem 11. *A closed Kähler manifold homeomorphic to $\mathbb{C}P^n$ is biholomorphic to it. If $n \leq 6$, the same is true with "homeomorphic" replaced by "homotopy equivalent".*

On the other hand, a fixed closed topological 4-manifold can support infinitely many distinct smooth structures each admitting a Kähler structure. Such examples be built using the so-called logarithmic transformations of elliptic surfaces (for details see [18]).

7.2 Low dimensions

Any complex curve admits a compatible Kähler metric (it is easy to see that on the open disc a complex structure determines Kähler metric up to conformal rescaling; on a general surface, pick metrics locally and rescale on the intersections if necessary).

In complex dimension 2, a closed complex surface is Kähler iff it has even first Betti number. This can be shown either via the classification of complex surfaces or more directly (the latter was first achieved independently by Buchdach and Lamari).

7.3 Hodge numbers

For Kähler manifolds, Hodge numbers are invariant under small deformations (see for example [19]).

Theorem 12. *Let $f : X \to B$ be a family of complex manifolds and assume that X_0 is Kähler for some $0 \in B$. Then for b in a neighbourhood of 0 in B, the Hodge numbers of X_b are the same as the Hodge numbers of X_0.*

This need not hold for large deformations, however. The following is a theorem due to Kotschick.

Theorem 13. *A rational linear combination of Hodge and Chern numbers of closed K ahler manifolds is a homeomorphism invariant in any dimension, or a diffeomorphism invariant in dimension $n = 2$, iff it reduces to a linear combination of the Betti numbers after perhaps adding a suitable combination of $\chi_p - Td_p$ (the Hodge numbers are considered modulo the Kähler symmetries.*

Here χ_p is $\Sigma_q(-1)^q h^{p,q}(X)$ and Td_p is the top-dimensional component of $Td(X)Ch(\Omega_X^p)$. This theorems roughly means that Hodge and Chern numbers are not topological invariants unless there is an obvious reason for it.

8 Smooth complex projective varieties

Among complex manifolds, the more well-studied class is that of closed complex submanifolds of complex projective space (note that by Chow's lemma, it can be identified with the class of submanifolds of complex projective space defined by a finite set of polynomial equations).

One of the advantages of the algebraic category is that one can prove results using the tools of algebraic geometry (for example, Hodge degeneration for smooth projective varieties can be established by mod p reduction [20]).

It is natural to ask what Kähler manifolds can be embedded into $\mathbb{C}P^n$. The following theorem, due to Kodaira, provides an answer.

Theorem 14. *A closed Kähler manifold can be holomophically embedded into complex projective space $\mathbb{C}P^n$ for some n if and only if its Kähler class lies in the image of the natural map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$.*

Corollary 1. *A closed complex manifold admitting a compatible Kähler structure and satisfying $h^{2,0} = 0$ can be holomorphically embedded into complex projective space.*

Proof. It can be shown that the set of Kähler classes is non-empty open in $H^{1,1}(M) \cap H^2(M, \mathbb{R})$. From Hodge symmetry, we get that $h^{0,2} = h^{2,0} = 0$. Therefore, $H^{1,1}(M) = H^2(M, \mathbb{C})$, so $H^{1,1}(M) \cap H^2(M, \mathbb{R}) = H^2(M, \mathbb{R})$. This means that the set of Kähler classes is open in $H^2(M, \mathbb{R})$ and, in particular, has non-trivial intersection with the dense set $H^2(M, \mathbb{Q})$. Picking some element of the intersection and rescaling it, we get a Kähler class in $H^2(M, \mathbb{Z})$. Now we are done by Kodaira embedding theorem. □

Kodaira also proved that every closed Kähler surface is a deformation of a projective Kähler surface. Apparently, this led Kodaira to conjecture that any closed Kaehler manifold is a deformation of a projective Kähler manifold;

however, in every complex dimension $n \geq 4$, Voisin has constructed examples of closed Kähler manifolds which do not have the homotopy type of a projective Kähler manifold. Kodaira's conjecture is still open in complex dimension 3.

Remark 8. *Though we know that there are obstructions at the homotopy type level for a closed Kähler manifold to admit a holomorphic embedding to projective space, no obstructions on the level of fundamental groups are known.*

Now we will consider the most well-studied class of projective varieties: Fano varieties.

8.1 Fano varieties

Recall that a line bundle L on a complex manifold M is called ample if there exists some $n, m > 0$ and a holomorphic embedding $\phi : M \to \mathbb{C}P^n$ such that $L^{\otimes m} = \phi^*(O(1))$, where $O(1)$ is the vector bundle of homogeneous polynomials of degree 1 on $\mathbb{C}P^n$.

Definition 17. *A complex manifold M is called Fano if its anticanonical bundle is ample.*

It can be shown that for a closed Fano manifold, $h^{0,n}(X) = 0$ for $n > 0$. Yau's theorem (described below) implies that a smooth complex variety admits Kähler metrics of positive Ricci curvature if and only if it is Fano. Myers theorem therefore tells us that the universal cover of a Fano manifold is compact, and so can only be a finite covering. Since the Todd genus $Td(X) = \sum(-1)^n h^{0,n}(X)$ of both X and its universal cover has to be equal to 1, and since the Todd genus is multiplicative under finite covers, we get that any Fano manifold is simply connected.

It is known that there are only finitely many deformation classes of smooth Fano varieties in each dimension. A Fano curve is isomorphic to the projective line so there is only one deformation class. Fano surfaces, also known as del Pezzo surfaces, come in 10 deformation classes. In complex dimension 3, there are 105 deformation classes. It can be shown easily that the number of deformation classes of smooth Fano varieties as a function of dimension has to be superpolynomial but the author is not aware of precise asymptotics.

Remark 9. *Note that a complex manifold with ample anticanonical bundle is automatically projective. It is natural to ask whether every closed symplectic manifold (M, ω) with $c_1(M) = \lambda[\omega]$ for some $\lambda > 0$ is Kähler/projective. McDuff has proven that that every closed 4-dimensional symplectic Fano manifold is a projective manifold. In real dimension 6, 8 or 10, it is not known whether there exist non-projective closed symplectic Fano's while in dimensions 12 and higher there are infinitely many non-simply connected symplectic Fano manifolds—in particular, they are not Kähler. See works of J. Fine and D. Panov for details.*

9 Kähler–Einstein manifold

A Kähler metric that is at the same time an Einstein metric will be referred to as KE metric.

9.1 Positive Einstein constant

It is known that on a closed complex manifold with ample anticanonical bundle any two KE metrics with Einstein constant 1 are related by a biholomorphism.

It is known that the Lie algebra of holomorphic vectors fields on a closed complex manifold admitting a Kähler–Einstein metric with positive Einstein constant is reductive. For the surfaces, the converse is true (using this, one can verify that out of 10 deformation classes of del Pezzo surfaces, surfaces in 2 deformation classes do not admit KE metric).

Remark 10. *There do exist closed complex manifolds that admit both a Kähler metric and an Einstein metric but no KE metric (see [22]).*

9.2 Zero Einstein constant

The following is a theorem due to Yau.

Theorem 15. *If M is a closed Kähler manifold with Kähler form ω, and R is any $(1,1)$-form representing $c_1(M)$, then there exists a unique Kähler metric $\tilde{g}$ on M with Kähler form $\tilde{\omega}$ such that $[\omega] = [\tilde{\omega}] \in H^2(M, \mathbb{R})$ and the Ricci form of $\tilde{\omega}$ is R.*

In particular, a compact Kähler manifold with a vanishing first real Chern class has a Ricci-flat Kähler metric class in the same Kähler class.

There are various definitions of Calabi–Yau manifolds (the one we use does not guarantee projectivity).

Definition 18. *A Calabi–Yau manifold is a Kähler manifold with trivial real first Chern class.*

By Yau's theorem, on a closed CY manifold in every Kähler class one has a Ricci-flat metric.

Remark 11. *Yau's theorem is an existence theorem. To the best of author's knowledge, there are no explicit examples of closed Ricci-flat Kähler–Einstein manifolds which are not locally isometric to Eucldiean space. However, there do exist explicit complete non-flat Ricci-flat Kähler metrics on non-compact spaces (e.g. Eguchi–Hanson metric).*

Remark 12. *Some people define Calabi–Yau manifold as a Kähler manifold whose canonical bundle is holomorphically trivial. This definition is much stronger than ours (though as shown by Bogomolov [23], some power of the canonical bundle of a CY manifold in our sense is holomorphically trivial). Note that there do exist non-Kähler complex manifolds whose canonical bundle is topologically* *trivial yet no power of the canonical bundle is holomorphically trivial (e.g. Hopf surface). See [24] for detailed discussion.*

In dimension 1, closed Calabi–Yau manifolds are elliptic curves.

In dimension 2, a closed CY manifold is either a K3 surface (i.e. a simply connected closed complex surface with trivial canonical bundle), a complex torus or a finite quotient of those two.

In general, one has the following theorem (Bogomolov–Beauville).

Theorem 16. *A closed CY manifold has a finite unbranched cover such that each factor is either:*

- *A complex torus with the standard metric;*
- *A closed Kähler manifold of dimension $n \geq 3$ with trivial canonical bundle and $h^{p,0} = 0$ for $0 < p < n$;*
- *a closed simply-connected Kähler manifold with a holomorphic symplectic form (unique up to rescaling).*

Note that the factors of the second type are projective by Kodaira embedding theorem (we refer to them as irreducible Calabi–Yau manifolds). The factors of the third type are generically not projective and are called *hyperkähler manifolds.*

It is conjectured that there are only finitely many topological types of irreducible Calabi–Yau manifolds of real dimension 6. It is known that bimeromorphic irreducible Calabi–Yau manifolds have equal Hodge numbers (see e.g. [29]).

Note that hyperkähler manifolds necessarily have real dimension divisible by 4.

It is known that for a closed hyperkähler manifold X there is an action of Lie algebra so_5 on $H^*(X, \mathbb{C})$ [25].

It is known that if two closed hyperkähler manifolds are bimeromorphic then they are deformation equivalent [26].

It is conjectured that there are finitely many topological types of closed hyperkähler manifolds in each dimension (see [28]). This conjecture is true in complex dimension 2 since any two smooth K3 surfaces are diffeomorphic. It is known that for closed hyperkähler fourfolds either $b_2 = 23$ or $3 \leq b_2 \leq 8$ holds (so in particular, the set of possible values of the second Betti number is finite).

9.3 Negative Einstein constant

A closed complex manifold with ample canonical bundle admits a unique Kähler–Einstein metric of Einstein constant -1.

The author is not aware of explicit examples of KE metrics with negative Einstein constant on closed manifolds which are not locally isometric to the standard hyperbolic metric. Constructing such examples appears to be difficult; for example, the obvious idea of considering restrictions of Fubini–Study metric will not work since a theorem of Hulin [30] says that a closed complex submanifold of $\mathbb{C}P^n$ such that the induced metric is Einstein has to have positive Einstein constant.

References

[1] On the minimal sum of Betti numbers of an almost complex manifold, arXiv preprint

[2] https://mathoverflow.net/questions/267929/proof-of-the-holomorphic-frobenius-theorem-in-voisins-book-on-hodge-theory-the/267930#267930

[3] Donaldson, S.: Polynomial invariant for smooth four-manifolds, Topology (1990), 257–315.

[4] Riemann Surfaces (Oxford Graduate Texts in Mathematics), S. Donaldson. Chapter 13

[5] Fundamental Groups of Compact Kähler Manifolds, J. Amoros et al.

[6] The Frölicher spectral sequence can be arbitrarily non-degenerate, arXiv preprint

[7] Complex Abelian varieties, Grundlehren der Mathematischen Wissenschaften, C. Birkenhake and H. Lange. Chapter 2

[8] Recherche de structures analytiques complexes sur certaines variétés. C. R. Acad. Sci. Paris, t. 236 (1953), 657–659. A. Blanchard

[9] The algebraic dimension of compact complex threefolds with vanishing second Betti number, arXiv preprint

[10] Ugarte, L. Geometriae Dedicata (2000) 81: 173.

[11] Group Actions on S^6 and complex structures on P_3, arXiv preprint

[12] Deligne, P., Griffiths, P., Morgan, J. et al. Invent Math (1975) 29: 245.

[13] Publications Mathématiques de l'IHÉS, Volume 47 (1977), p. 269-331.

[14] Geometry & Topology 22 (2018) 2115–2144.

[15] Chal Benson, Carolyn S. Gordon, Kähler and symplectic structures on nilmanifolds, Topology, Volume 27, Issue 4, 1988, 513-518.

[16] Two remarks on Calabi-Yau Moishezon threefolds, J. reine angew. Math. 452 (1994).

[17] Expo. Math. 35 (2017), no.1, 1-12.

[18] Notes on Seiberg-Witten Theory, L. Nicolaescu. Corollary 3.3.23

[19] Hodge Theory and Complex Algebraic Geometry, I (Cambridge Studies in Advanced Mathematics)

[20] Deligne, P., and Illusie, L. "Relevements modulo p^2 et decomposition du complexe de de Rham" Inventiones mathematicae 89 (1987): 247-270.

[21] Mathematical Research Letters, Volume 2 (1995), Number 6. Bounding curves in algebraic surfaces by genus and Chern numbers.

[22] On Conformally Kaehler, Einstein Manifolds, arXiv preprin

[23] F. A. Bogomolov, Kähler manifolds with trivial canonical class, Izv. Akad. Nauk SSSR Ser. Mat., 1974, Volume 38, Issue 1, 11–21.

[24] Contemp. Math. 644 (2015), 261-277.

[25] M. S. Verbitsky, "Action of the Lie algebra $SO(5)$ on the cohomology of a hyperkähler manifold", Funktsional. Anal. i Prilozhen., 24:3 (1990), 70–71.

[26] Huybrechts, D. Math. Ann. (2003) 326: 499.

[27] Salamon, D. Acta Math Vietnam. (2013) 38: 123.

[28] Holomorphic symplectic geometry: a problem list, arXiv preprint

[29] T. Ito, Birational smooth minimal models have equal Hodge numbers in all dimensions, Calabi-Yau varieties and mirror symmetry (Toronto, ON, 2001), 183–194, Fields Inst. Commun. 38, Amer. Math. Soc., Providence, RI, 2003.

[30] Hulin, D. J Geom Anal (2000) 10: 525.